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THE THOULESS FORMULA FOR RANDOM NON-HERMITIAN JACOBI MATRICES

BY

ILYA YA. GOLDSHEID

School of Mathematical Sciences, Queen Mary, University of London
London E1 4NS, U.K.
e-mail: I.Goldsheid@qmul.ac.uk

AND

BORIS A. KHORUZHENKO

School of Mathematical Sciences, Queen Mary, University of London
London E1 4NS, U.K.
and
Department of Mathematical Sciences, Brunel University
London UB8 3PH, U.K.
e-mail: B.Khoruzhenko@qmul.ac.uk

ABSTRACT

Random non-Hermitian Jacobi matrices J_n of increasing dimension n are considered. We prove that the normalized eigenvalue counting measure of J_n converges weakly to a limiting measure μ as $n \to \infty$. We also extend to the non-Hermitian case the Thouless formula relating μ and the Lyapunov exponent of the second-order difference equation associated with the sequence J_n. The measure μ is shown to be log-Hölder continuous. Our proofs make use of (i) the theory of products of random matrices in the form first offered by H. Furstenberg and H. Kesten in 1960 [8], and (ii) some potential theory arguments.

1. Introduction

Let a_j, b_j, and c_j be three given sequences of complex numbers. Consider the second-order difference equation for f

$$a_j f_{j-1} + b_j f_j + c_j f_{j+1} = z f_j, \quad j = 1, 2, \ldots \tag{1.1}$$

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This equation can be also written as

$$(1.2) \qquad \begin{pmatrix} f_{j+1} \\ f_j \end{pmatrix} = g_j \begin{pmatrix} f_j \\ f_{j-1} \end{pmatrix}, \quad j = 1, 2, \ldots, \quad \text{where } g_j = \begin{pmatrix} \frac{z-b_j}{c_j} & \frac{-a_j}{c_j} \\ 1 & 0 \end{pmatrix}.$$

Denote by $f_j(z)$ the solution of (1.1) satisfying the initial condition $f_0 = 0$, $f_1 = 1$. In terms of the transfer matrix $S_n(z) = g_n \cdot \ldots \cdot g_1$,

$$(1.3) \qquad \begin{pmatrix} f_{n+1}(z) \\ f_n(z) \end{pmatrix} = S_n(z) \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Obviously, $f_{n+1}(z)$ is a polynomial in z of degree n,

$$(1.4) \qquad f_{n+1}(z) = k_n \prod_{l=1}^{n} (z - z_l), \quad k_n = \prod_{j=1}^{n} 1/c_j.$$

Its roots $z_1, \ldots, z_n$ are the eigenvalues of the tridiagonal (Jacobi) matrix

$$(1.5) \qquad J_n = \begin{pmatrix} b_1 & c_1 & & & \\ a_2 & b_2 & c_2 & & \\ & \ddots & \ddots & \ddots & \\ & & a_{n-1} & b_{n-1} & c_{n-1} \\ & & & a_n & b_n \end{pmatrix}.$$

In this paper we are concerned with the limiting distribution of the eigenvalues of J_n as $n \to \infty$ for random a_j, b_j, and c_j.

If all b_j are real and $c_j = a_{j+1}^*$ for all j, the matrices J_n are Hermitian. The eigenvalue distribution of such matrices was studied extensively in the past in the context of the Anderson model; see, e.g., [19, 3]. In this case, the eigenvalues are always real and there are several ways to prove that the normalized eigenvalue counting measure of J_n converges to a limiting measure as $n \to \infty$. None of these proofs works in the non-Hermitian case and little is known about the limiting eigenvalue distribution of random non-Hermitian Jacobi matrices; however, see [5, 15].

Our interest in such matrices is partly motivated by non-Hermitian quantum mechanics of Hatano and Nelson [13, 14] which, in one dimension, leads to equation (1.1) with the coefficients a_j, b_j, and c_j chosen randomly from the special class defined by the restrictions

$$(1.6) \qquad b_j \in \mathbb{R} \quad \text{and} \quad a_{j+1}^*/c_j > 0 \quad \text{for all } j.$$

In this class the Liouville substitution[1] reduces equation (1.1) to the symmetric

1 $f_j = \theta_j \psi_j$, where $\theta_1 = 1$ and $\theta_k = (\prod_{j=1}^{k-1} a_{j+1}^*/c_j)^{1/2}$ for $k \geq 2$.

equation

$$s_{j-1}^*\psi_{j-1} + b_j\psi_j + s_j\psi_{j=1} = z\psi_j \tag{1.7}$$

where $s_j = c_j(a_{j+1}^*/c_j)^{1/2}$. However, the situation here is much richer than in the Hermitian case as the choice of boundary conditions to accompany equation (1.1) has a profound effect on the spectrum of the associated Jacobi matrix. If the Dirichlet boundary conditions, $f_0 = 0$ and $f_{n+1} = 0$, are chosen then the corresponding Jacobi matrix is J_n (1.5). Since the Dirichlet boundary conditions are preserved by the Liouville transformation, the spectrum of J_n is real provided the coefficients (a_j, b_j, c_j) belong to the Hatano–Nelson class (1.6). On the other hand, if one imposes the periodic boundary conditions, $f_0 = f_n$ and $f_1 = f_{n+1}$, then the spectrum of the corresponding Jacobi matrix turns out to be complex. This is not surprising, of course, as the Liouville substitution transforms the periodic boundary conditions for f into highly asymmetric boundary conditions for ψ. What is surprising, however, is that in the limit $n \to \infty$ the complex eigenvalues lie on analytic curves [10] and are regularly spaced even if the coefficients in equation (1.1) are chosen randomly [11]. These effects are specific to the Hatano–Nelson class and the proofs and analysis of the limiting eigenvalue distribution given in [10, 11] exploit the relation between equations (1.1) and (1.7). Of course, in the general case of arbitrary coefficients no such relation exists and one requires a different approach in order to investigate the eigenvalue distribution of J_n. We develop such an approach in the present paper.

Throughout this paper we assume that:

A1. $\{(a_j, b_j, c_j)\}_{j=1}^{\infty}$ is a sequence of i.i.d. random vectors.

A2. For some $\delta > 0$, $E[|a_j|^\delta + |a_j|^{-\delta} + |b_j|^\delta + |c_j|^\delta + |c_j|^{-\delta}] < \infty$.

A3. The support of the probability distribution of the random vector (a_1, b_1, c_1) contains at least two different points (a, b, c) and (a', b', c').

If all mass of the probability distribution of (a_j, b_j, c_j) is concentrated at one point (a, b, c), then of course we have a tridiagonal matrix with constant diagonals. This is a particular case of Töplitz matrices. The eigenvalue distribution of non-Hermitian Töplitz matrices was extensively studied in the past; see, e.g., survey [22].

Our main result expresses the limiting distribution of the eigenvalues of J_n in terms of the (upper) Lyapunov exponent

$$\gamma(z) = \lim_{n\to\infty} \frac{1}{2n} \log[|f_{n+1}(z)|^2 + |f_n(z)|^2]$$

of equation (1.1). It is well known that (for every complex z) the above limit exists with probability one and is nonrandom. This follows from Oseledec's multiplicative ergodic theorem [18]. A more subtle fact is that in our case $\gamma(z)$ can be calculated using the well known Furstenberg formula [7], and moreover,

$$\gamma(z) = \lim_{n\to\infty} \frac{1}{n} E \log ||S_n(z)||, \quad z \in \mathbb{C}. \tag{1.8}$$

The function $\gamma(z)$ is subharmonic in the entire complex plane [4] and bounded from below,

$$\gamma(z) \geq \frac{1}{2} E \log |a_1/c_1| \quad \text{for all } z. \tag{1.9}$$

This inequality easily follows from $\det S_n(z) = \prod_{j=1}^{n} a_j/c_j$. The subharmonicity implies that $\Delta\gamma$, where Δ is the distributional Laplacian in variables $\operatorname{Re} z$ and $\operatorname{Im} z$, defines a measure on $\mathbb{C}$; see, e.g., [17]. Our main result is as follows.

THEOREM 1.1: *Let μ_n be the normalized eigenvalue counting measure of J_n, i.e., $\mu_n = \frac{1}{n}\sum_{l=1}^{n} \delta_{z_l}$, where $z_1, \ldots, z_n$ are the eigenvalues of J_n. Then:*

(a) *With probability one, μ_n converges weakly to $\mu = \frac{1}{2\pi}\Delta\gamma$ as $n \to \infty$.*

(b) *(Thouless formula) For every $z \in \mathbb{C}$,*

$$\gamma(z) = \int_{\mathbb{C}} \log|w - z| d\mu(w) - E \log|c_1|. \tag{1.10}$$

(c) *The limiting eigenvalue counting measure μ is log-Hölder continuous. More precisely, for any $B_{z_0,\delta} = \{z : |z - z_0| \leq \delta\}$, $0 < \delta < 1$,*

$$\mu(B_{z_0,\delta}) \leq \frac{C(z_0, \delta)}{\log \frac{1}{\delta}}, \tag{1.11}$$

where $C(z_0, \delta) \to 0$ as $\delta \to 0$.

We deduce Theorem 1.1 from Theorem 1.2 which is of independent interest in the context of second order difference equations.

THEOREM 1.2: *With probability one,*

$$\lim_{n\to\infty} \frac{1}{n} \log |f_{n+1}(z)| = \gamma(z) \tag{1.12}$$

for almost all z with respect to Lebesgue measure on $\mathbb{C}$.

In the real case, i.e., when the sequences a_j, b_j and c_j are real *and* the spectral parameter z is real, Theorem 1.2 can be deduced from the Law of Large Numbers for the coefficients of the product of random matrices which was proved in [12]. Our proof of Theorem 1.2 is different from the one given in

[12]. It is well adapted to our (complex valued) case, very short and makes our paper self-contained.

In the Hermitian case, (1.12) and the Thouless formula (1.10) follow directly from the fact that μ_n converges weakly to a limiting measure μ [1, 6] and the latter can be established without use of products of random matrices and by more elementary means. We would like to emphasize that in the non-Hermitian case we follow the opposite direction route: the weak convergence of μ_n and the Thouless formula are deduced from (1.12). To this end we make use of the relation

$$\mu_n = \frac{1}{2\pi n}\Delta \log|f_{n+1}|, \tag{1.13}$$

where the equality is to be understood in the sense of distribution theory. Relation (1.13) is well known in function theory. It holds for an arbitrary polynomial of degree n and can be easily derived with the help of the Gauss–Green formula. In this general setup it was shown by Widom [21, 22] that if the measures μ_n for all n are supported inside a bounded region and in the limit $n \to \infty$ the function $p_n(z) = \int_{\mathbb{C}} \log|z-w|d\mu_n(w)$ converges to a limiting function $p(z)$ almost everywhere in the complex plane, then μ_n converges weakly to $\mu = \frac{1}{2\pi}\Delta p$. We shall need the following simple extension of this result to the case when the supports of μ_n are not necessarily bounded.

Let A_n be a (deterministic) sequence of square matrices of increasing dimension n, and

$$p_n(z) = \frac{1}{n}\log|\det(A_n - zI_n)| = \int_{\mathbb{C}} \log|w-z|d\mu_n(w),$$

where I_n is the $n \times n$ identity matrix and $\mu_n = \frac{1}{2\pi}\Delta p_n$ is the normalized eigenvalue counting measure of A_n. Define

$$\tau_R = \limsup_{n\to\infty} \int_{|w|\geq R} \log|w|d\mu_n(w), \quad R \geq 1. \tag{1.14}$$

PROPOSITION 1.3: *Assume that there is a function p: $\mathbb{C} \to [-\infty, +\infty)$ such that $p_n(z) \to p(z)$ as $n \to \infty$ almost everywhere in $\mathbb{C}$. If $\tau_1 < +\infty$ then it follows that p is locally integrable, $\mu = \frac{1}{2\pi}\Delta p$ is a unit mass measure,*

$$\int_{|w|\geq 1} \log|w|d\mu(w) \leq \tau_1 < +\infty, \tag{1.15}$$

and the sequence of measures μ_n converges weakly to μ as $n \to \infty$. If, in addition, $\lim_{R\to\infty} \tau_R = 0$ then we also have that

$$p(z) = \int_{\mathbb{C}} \log|w-z|d\mu(w). \tag{1.16}$$

Remark: In view of (1.15), the integral on the RHS in (1.16) is a locally integrable function of z taking values in $[-\infty, +\infty)$.

For the sake of completeness, we give a proof of this Proposition in Appendix A.

In order to estimate the tails of eigenvalue distributions as required in the above Proposition 1.3, we use the following inequalities[2]:

$$\tau_1 \le \limsup_{n\to\infty} \frac{1}{2n} \log\det(I_n + A_n A_n^*), \tag{1.17}$$

and for any $R > 1$ and $\delta > 0$,

$$\tau_R \le \frac{1}{\log^\delta R} \limsup_{n\to\infty} \frac{1}{2^{1+\delta} n} \operatorname{tr} \log^{1+\delta}(I_n + A_n A_n^*). \tag{1.18}$$

These inequalities can be derived with the help of Weyl's Majorant Theorem; for details of the derivation, see Appendix B.

Let us now return to the random Jacobi matrices J_n. Straightforward but tedious calculations show[3] that

$$\frac{1}{n} \operatorname{tr} \log^{1+\delta}(I_n + J_n J_n^*) \le \frac{\alpha}{n} \sum_{j=1}^{n} \log^{1+\delta}(1 + \beta|\mathbf{v}_j|^2), \quad \text{where } \mathbf{v}_j = (a_j, b_j, c_j),$$

for some $\alpha, \beta > 0$ independent of $\mathbf{v}_j$'s and n. Therefore, if the random sequence $\mathbf{v}_j$ is stationary and

$$E \log^{1+\delta}[1 + |\mathbf{v}_1|^2] < \infty \quad \text{for some } \delta > 0, \tag{1.19}$$

then the Ergodic Theorem asserts that with probability one the limits in (1.17), (1.18) are finite which implies $\tau_1 < \infty$ and $\lim_{R\to\infty} \tau_R = 0$, as required in Proposition 1.3. The assumptions of stationarity and (1.19) are less restrictive than assumptions A1–A3. However, we are only able to prove Theorem 1.2 (which is the main ingredient in our proof of Theorem 1.1) under these more restrictive assumptions.

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2 Note that $\log\det(I_n + A_n A_n^*) = \operatorname{tr}\log(I_n + A_n A_n^*)$.

3 For any Hermitian matrix $H = \|H_{jk}\|_{j,k=1}^n$ we have $H \le D = \operatorname{diag}(d_1, \ldots, d_n)$ with $d_j = \sum_{k=1}^n |H_{jk}|$, $j = 1, \ldots, n$. Therefore, if f is a nondecreasing function then, by the Courant–Fisher minimax principle, $\operatorname{tr}\log f(H) \le \operatorname{tr} f(D) = \sum_{j=1}^n f(d_j)$.

2. Products of random matrices

Our proof of Theorem 1.2 makes use of several facts from the theory of products of random 2×2 matrices. We list these facts below (Propositions 2.1–2.3).

Let ν be a probability distribution on the group $Gl(2,\mathbb{C})$ of invertible complex 2×2 matrices and g_k be an infinite sequence of independent samples from this distribution.

As before, $S_n = g_n \cdot \ldots \cdot g_1$ for $n = 1, 2, \ldots$. By $P(\mathbb{C}^2)$ we denote the projective space on which every non-degenerate matrix g acts in a natural way. Let κ be a probability measure on $P(\mathbb{C}^2)$. We say that g preserves κ if $\kappa(g^{-1}.B) = \kappa(B)$ for any Borel set B (here $g.x$ is the result of the action of g on $x \in P(\mathbb{C}^2)$). By G_ν we denote the closure of the subgroup of $Gl(2,\mathbb{C})$ generated by all matrices belonging to the support of ν. We say that G_ν preserves κ if κ is preserved by every $g \in G_\nu$.

PROPOSITION 2.1: *Let $\lambda_1^{(n)} \geq \lambda_2^{(n)}$ be the singular values of S_n. If*

$$E \log ||g|| \text{ and } E \log |\det g| \text{ are both finite,} \tag{2.1}$$

then with probability one the following limits

$$\lim_{n\to\infty} \frac{1}{n} \log \lambda_j^{(n)} = \gamma_j, \quad j = 1, 2, \tag{2.2}$$

exist and are nonrandom.

The limiting values γ_1 and γ_2 are called the Lyapunov exponents of the sequence S_n.

PROPOSITION 2.2: *If, in addition to condition (2.1), no measure κ is preserved by G_ν, then the Lyapunov exponents of the sequence S_n are distinct, i.e., $\gamma_1 > \gamma_2$.*

PROPOSITION 2.3: *If condition (2.1) is satisfied and no measure κ is preserved by G_ν, then:*

(i) *For any unit vector x the probability is one that*

$$\lim_{n\to\infty} \frac{1}{n} \log ||S_n x|| = \gamma_1. \tag{2.3}$$

(ii) *If in addition $E(||g||^\delta + ||g^{-1}||^\delta) < \infty$ for some $\delta > 0$, then for any positive ε there is a constant $\rho(\varepsilon) > 0$ such that uniformly in x, $||x|| = 1$,*

$$Prob(|\log ||S_n x|| - n\gamma_1| \geq \varepsilon n|) \leq e^{-n\rho(\varepsilon)}. \tag{2.4}$$

Remarks: 1. As all norms in $\mathbb{C}^2$ are equivalent, the choice of norm in (2.3) and (2.4) is not important. However, it is convenient to deal with the standard Euclidian norm.

2. Propositions 2.1–2.3 are well known in the classical case of the real matrices; see, e.g., [18, 2] for proofs of Propositions 2.1 and 2.3 and [7, 20] for proofs of Proposition 2.2. For complex matrices, Propositions 2.1 and 2.3 are proved in the same way as in [18, 2]. However, the proof of Proposition 2.2 is somewhat different from that given in [7, 20]. We shall now discuss the necessary changes which would allow the interested reader to reconstruct the proof in question simply by examining the one in [20]. Namely, the main ingredient of this proof is the fact that the mapping $g \mapsto T_g$, where

$$(T_g f)(x) = f(g^{-1}x)||g^{-1}x||^{-m/2},$$

defines a unitary representation of the group $SL(m, \mathbb{R})$ in Hilbert space $L_2(\mathcal{S}_m, dl)$ with dl being the natural Lebesgue measure on the unit sphere $\mathcal{S}_m \in \mathbb{R}^m$. (Obviously, we are interested in the case when $m = 2$.)

In the case of the complex space the representation is defined by

$$(T_g f)(x) = f(g^{-1}x)||g^{-1}x||^{-m},$$

in Hilbert space $L_2(\mathcal{S}_m, dl)$ with dl being again the natural Lebesgue measure on the unit sphere $\mathcal{S}_m \in \mathbb{C}^m$. After that the proof proceeds in the way suggested in [20].

3. Proofs of Theorems 1.1 and 1.2

In order to be able to apply Propositions 2.1–2.3, we have to verify that under assumptions A1–A3 our matrices g_j defined in (1.2) satisfy the conditions of these Propositions.

It is apparent that assumption A2 guarantees that condition (2.1) is satisfied and $E(||g||^\delta + ||g^{-1}||^\delta) < \infty$. It remains to check that assumption A3 implies that no measure κ is preserved by G_ν (here ν is the measure induced on the group of matrices by the distribution of (a_1, b_1, c_1)). To this end we note that if

$$g = \begin{pmatrix} \frac{z-b}{c} & \frac{-a}{c} \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad g' = \begin{pmatrix} \frac{z-b'}{c'} & \frac{-a}{c'} \\ 1 & 0 \end{pmatrix}$$

then

$$gg'^{-1} = \begin{pmatrix} \frac{c'a}{a'c} & \frac{z-b}{c} - \frac{(z-b')a}{a'c} \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad g'^{-1}g = \begin{pmatrix} 1 & 0 \\ \frac{z-b'}{a'} - \frac{(z-b)c'}{ca'} & \frac{c'a}{a'c} \end{pmatrix}.$$

It remains to check that for almost all z the group G generated by the matrices g, g' is rich enough in the sense that no measure is preserved by all matrices of this group. The main idea is as follows. For a "typical" z we construct two matrices, say B and D, from G such that the eigenvalues of B are of different moduli. It is easy to see then that the only measure preserved by all matrices of the form B^n, $-\infty < n < \infty$ is the one supported by the lines in $P(\mathbb{C}^2)$ generated by the eigenvectors of B. The matrix $D \in G$ is then chosen so that its action on $P(\mathbb{C}^2)$ does not preserve these lines, which means that the measure in question does not exist. We would like to emphasize that the presence of the parameter z plays a crucial role in this situation.

More precisely, if z is such that

$$2\arg(z-b) \neq \arg(ac),$$

then the matrix g has eigenvalues with different moduli. In other words, the moduli are different if z does not belong to a certain half line. The g' then plays the role of D (once again when z lies outside of certain curves). This statement can be checked by direct calculation and is sufficient for our purposes.

However, in some important cases much more precise statements can be made. In particular, if $c'a/a'c = 1$ then each of triangular matrices (gg'^{-1}) and $g'^{-1}g$ is non-trivial for all but maybe two values of z and a similar idea applies; see [2] page 213.

Now we are in a position to apply Propositions 2.1–2.3. For any two non-zero vectors x and y, define

$$d(x,y) = \sqrt{1 - \frac{|(x,y)|^2}{(x,x)(y,y)}},$$

where $(\cdot,\cdot)$ is the scalar product in $\mathbb{C}^2$. The function $d(x,y)$ is the natural angular distance between x and y on the projective space $P(\mathbb{C}^2)$.

The following Lemma is the key element in the proof of Theorem 1.2. (In this Lemma and thereafter the abbreviation a.s. refers to the probability measure, i.e., any equality with the letters a.s. above it holds with probability one.)

Lemma 3.1: *Suppose that the conditions of Propositions 2.1–2.3 are satisfied. If y_n is a sequence of random unit vectors in $\mathbb{C}^2$ such that*

$$||S_n y_n|| = e^{n\gamma_2+\epsilon_n}, \quad \text{where } \epsilon_n \overset{\text{a.s.}}{=} o(n) \text{ as } n \to \infty, \tag{3.1}$$

then for any fixed unit vector x and any $\delta > 0$ there is a constant $r(x,\delta) > 0$ such that

$$Prob\{d(x,y_n) \le e^{-n\delta}\} \le e^{-nr(x,\delta)} \tag{3.2}$$

for all sufficiently large n.

Proof: For any n, one can always find two orthogonal unit vectors u_n and v_n such that $S_n^* S_n u_n = \lambda_1^{(n)}$ and $S_n^* S_n v_n = \lambda_2^{(n)}$. In view of Proposition 2.1,

$$||S_n u_n|| = e^{n\gamma_1 + \epsilon_n'} \quad \text{and} \quad ||S_n v_n|| = e^{n\gamma_2 + \epsilon_n''}, \quad \text{where } \epsilon_n', \epsilon_n'' \overset{\text{a.s.}}{=} o(n).$$

Obviously, the sequence v_n satisfies condition (3.1) and we first prove the large deviation estimate (3.2) for this sequence.

Let x be a fixed unit vector. Then $x = (x, u_n)u_n + (x, v_n)v_n$ for every n, and, since $|(x, u_n)| = d(x, v_n)$ and $|(x, v_n)| \le 1$, we have that

$$||S_n x|| \le d(x, v_n)||S_n u_n|| + ||S_n v_n||.$$

Therefore, if $d(x, v_n) \le e^{-n\delta}$ then

$$\log ||S_n x|| \le n\gamma_1 + \log(e^{-n\delta + \epsilon_n'} + e^{-n(\gamma_1 - \gamma_2) + \epsilon_n''}),$$

and hence with probability one,

$$\log ||S_n x|| - n\gamma_1 \le -n \min(\delta, \gamma_1 - \gamma_2) + o(n).$$

It follows now from Proposition 2.3 that

$$Prob\{d(x, v_n) \le e^{-n\delta}\} \le e^{-nr(x,\delta)} \tag{3.3}$$

for some $r(x, \delta)$ and all $n > n_0$, where n_0 depends on the matrices S_n, and also on x and δ.

Now, let y_n be an arbitrary sequence of random unit vectors satisfying condition (3.1), and let $y_n^\perp$ be a sequence of unit vectors orthogonal to y_n, i.e., $(y_n, y_n^\perp) = 0$ for all n. Obviously, $d(u_n, y_n^\perp) = |(u_n, y_n)|$ and, since $S_n^* S_n u_n = e^{2n\gamma_1 + 2\epsilon_n'} u_n$, we have that with probability one,

$$d(u_n, y_n^\perp) = e^{-2n\gamma_1 + o(n)} |(S_n u_n, S_n y_n)| \le e^{-n(\gamma_1 - \gamma_2) + o(n)}.$$

It is then apparent that $d(v_n, y_n)$ is also exponentially small for large n and therefore the large deviation estimate (3.2) for y_n follows from (3.3). ∎

Proof of Theorem 1.2: Let

$$x = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad y_n = \begin{pmatrix} f_{n+1}(z) \\ f_n(z) \end{pmatrix}, \quad n = 1, 2, \ldots.$$

Then

$$d^2(x, y_n) = \frac{|f_{n+1}(z)|^2}{|f_{n+1}(z)|^2 + |f_n(z)|^2} = \frac{|f_{n+1}(z)|^2}{||y_n||^2},$$

and therefore

$$\frac{1}{n}\log|f_{n+1}(z)| = \frac{1}{n}\log d(x, y_n) + \frac{1}{n}\log||y_n||. \tag{3.4}$$

In view of (1.3) and Proposition 2.3(i),

$$\lim_{n\to\infty}\frac{1}{n}\log||y_n|| \overset{\text{a.s.}}{=} \gamma_1(z), \tag{3.5}$$

where $\gamma(z)$ is the upper Lyapunov exponent of the sequence of transfer matrices $S_n(z)$. On the other hand, $S_n^{-1}(z)y_n = (1,0)^T$, and therefore

$$\lim_{n\to\infty}\frac{1}{n}\log\frac{||S_n^{-1}(z)y_n||}{||y_n||} \overset{\text{a.s.}}{=} -\gamma_1(z).$$

It follows now from Lemma 3.1 (applied to the matrices $S_n^{-1}(z)$ and the vectors x and $y_n/||y_n||$[4]) and the Borel–Cantelli Lemma that

$$\lim_{n\to\infty}\frac{1}{n}\log d(x, y_n) \overset{\text{a.s.}}{=} 0.$$

Therefore, in view of (3.4) and (3.5), for any fixed z the probability is one that

$$\lim_{n\to\infty}\frac{1}{n}\log|f_{n+1}(z)| = \gamma_1(z). \tag{3.6}$$

But then the probability is one that (3.6) holds almost everywhere in the complex plane. This follows from the Fubini Theorem. Our proof of Theorem 1.2 is complete.

Proof of Theorem 1.1: As explained in the Introduction, under assumptions A1–A3, the probability is one that $\tau_1 \leq C$ for some non-random $C < +\infty$ and $\lim_{R\to\infty}\tau_R = 0$. Therefore, parts (a) and (b) of Theorem 1.1 follow immediately from Theorem 1.2 by the way of Proposition 1.3.

The log-Hölder continuity of μ is a corollary of the Thouless formula and the fact that the Lyapunov exponent $\gamma(z)$ is bounded from below. This is done very much in the same way as in the Hermitian case; see [4]. The stronger property of Hölder continuity of μ is well known in the Hermitian case (see [3] for detailed

4 If γ_1 and γ_2 are the Lyapunov exponents of a sequence S_n, then the sequence S_n^{-1} has the Lyapunov exponents $-\gamma_2$ and $-\gamma_1$.

discussion) but it is not our intention to go that far in the non-Hermitian case in this short communication.

To prove (1.11), we first note that the integral $\int_{\mathbb{C}} \log|w-z| d\mu(w)$ converges absolutely for every z. Indeed, it follows from (1.15) that

$$\int_{|w-z|\geq 1} \log|w-z| d\mu(w) < +\infty,$$

and this inequality together with the Thouless formula and the lower bound (1.9) imply that

$$\int_{|w-z|\leq 1} |\log|w-z|| d\mu(w) < +\infty$$

as well. Therefore,

$$C(z,\delta) := \int_{|w-z|\leq \delta} |\log|w-z|| d\mu(w) \to 0 \quad \text{as } \delta \to 0. \tag{3.7}$$

Obviously, for $\delta < 1$,

$$C(z,\delta) = \int_{|w-z|\leq \delta} \log(1/|w-z|) d\mu(w) \geq \frac{\mu(B_{z,\delta})}{\log(1/\delta)},$$

and part (c) of Theorem 1.1 follows. Our proof of Theorem 1.1 is now complete.

A. Appendix

Proof of Proposition 1.3: The local integrability of $\log|z|$ and the condition $\tau_1 < +\infty$ imply that the functions $p_n(z)$ are uniformly integrable on bounded sets in $\mathbb{C}$. It follows from this that $p(z)$ is locally integrable and $p_n \to p$ as $n \to \infty$ in $D'(\mathbb{C})$, the space of Schwartz distributions in $\mathbb{C}$. Since Δ is continuous on distributions, we also have that $\Delta p_n \to \Delta p$ in $D'(\mathbb{C})$. Obviously $\Delta p \geq 0$, hence Δp is defined by a measure; see, e.g., [16]. As any sequence of measures converging as distributions must converge weakly, we conclude that $\mu_n = \frac{1}{2\pi}\Delta p_n \to \mu = \frac{1}{2\pi}\Delta p$ weakly as measures.

For any $R > 1$,

$$\int_{|w|\geq R} d\mu_n(w) \leq \frac{1}{\log|R|} \int_{|w|\geq 1} \log|w| d\mu_n(w).$$

Therefore the inequality $\tau_1 < +\infty$ implies that the sequence of measures μ_n is tight, and hence cannot lose mass. As each of μ_n has unit mass, so has the limiting measure μ.

It follows from the weak convergence of μ_n to μ and (1.14) that

$$\int_{1\le|w|\le R}\log|w|d\mu(w)\le\lim_{n\to\infty}\int_{1\le|w|\le 2R}\log|w|d\mu_n(w)\le\tau_1$$

for any $R>1$. This implies (1.15). Similarly, if $\lim_{R\to\infty}\tau_R=0$ then

$$\lim_{R\to\infty}\int_{|w|\ge R}\log|w|d\mu(w)=0. \tag{A.1}$$

It remains to prove relation (1.16). It will suffice to show that

$$p_n\to\int_{\mathbb{C}}\log|w-\cdot|d\mu(w)\quad\text{in } D'(\mathbb{C}) \tag{A.2}$$

when $n\to\infty$. Let $\psi(z)$ be a continuous function with bounded support. Then

$$\int_{\mathbb{C}}p_n(z)\psi(z)d^2z=\int_{\mathbb{C}}g(w)d\mu_n(w)$$

with

$$g(w)=\int_{\mathbb{C}}\psi(z)\log|w-z|d^2z.$$

The function g is continuous and $g(w)=O(\log|w|)$ when $|w|\to\infty$. Assume now that $\lim_{R\to\infty}\tau_R=0$. Then

$$\lim_{R\to\infty}\limsup_{n\to\infty}\int_{|w|\ge R}|g(w)|d\mu_n(w)=0,$$

and

$$\lim_{R\to\infty}\int_{|w|\ge R}|g(w)|d\mu(w)=0$$

because of (A.1). It now follows from the weak convergence of μ_n to μ that

$$\lim_{n\to\infty}\int_{\mathbb{C}}g(w)d\mu_n(w)=\int_{\mathbb{C}}g(w)d\mu(w).$$

Therefore

$$\lim_{n\to\infty}\int_{\mathbb{C}}p_n(z)\psi(z)d^2z=\int_{\mathbb{C}}g(w)d\mu(w)=\int_C\left\{\int_{\mathbb{C}}\log|w-z|d\mu(z)\right\}\psi(z)d^2z,$$

and (A.2) follows.

B. Appendix

Derivation of inequalities (1.17) and (1.18): Let $z_1, \ldots, z_n$ and $s_1, \ldots, s_n$ be respectively the eigenvalues and singular values of A_n labeled so that $|z_1| \geq |z_2| \geq \cdots \geq |z_n|$ and $s_1 \geq s_2 \geq \cdots \geq s_n$. Weyl's Majorant Theorem, see [9], page 39, asserts that

$$\sum_{j=1}^{m} F(|z_j|) \leq \sum_{j=1}^{m} F(s_j), \quad m = 1, 2, \ldots, n,$$

for any function $F(t)$ $(0 \leq t < \infty)$ such that $F(e^x)$ is convex on $\mathbb{R}$. Obviously, the function $\log^{1+\delta}(t)$ satisfies this requirement for $\delta \geq 0$, and therefore

$$\int_{|w|\geq 1} \log^{1+\delta}|w| d\mu_n(w) = \frac{1}{n} \sum_{|z_j|\geq 1} \log^{1+\delta}|z_j| \leq \frac{1}{n} \sum_{|z_j|\geq 1} \log^{1+\delta}|z_j|$$
$$\leq \frac{1}{n} \sum_{j=1}^{m} \log^{1+\delta} s_j$$

where m is the number of eigenvalues of A_n such that $|z_j| \geq 1$. Obviously,

$$\sum_{j=1}^{m} \log s_j = \frac{1}{2^{1+\delta}} \sum_{j=1}^{m} \log^{1+\delta}(s_j^2) \leq \frac{1}{2^{1+\delta}} \sum_{j=1}^{n} \log^{1+\delta}(1 + s_j^2)$$
$$= \frac{1}{2^{1+\delta}} \operatorname{tr} \log^{1+\delta}(I_n + A_n A_n^*),$$

and therefore

$$\int_{|w|\geq 1} \log^{1+\delta}|w| d\mu_n(w) \leq \frac{1}{2^{1+\delta} n} \operatorname{tr} \log^{1+\delta}(I_n + A_n A_n^*), \quad \delta \geq 0,$$

which implies (1.17) and (1.18).

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